On the Medians of a Closed Convex Polygon.

By Arnold Emch.

§ 1. Introduction.

In an article which appeared in this Journal* I have proved that all *medians* of a closed convex analytic curve, defined as loci of the mid-points of chords of all systems of parallel chords, are continuous and analytic curves. With each pair of orthogonal directions σ and τ are associated two medians M_σ and M_τ which always intersect in one and only one point. It is assumed that the tangent changes continuously as the point of tangency moves continuously on the curve. With every pair (σ, τ) a rhomb $A_1 A_2 A_3 A_4$ is associated whose diagonals $A_1 A_3$ and $A_2 A_4$ are parallel to σ and τ and whose vertices lie on the given curve. This fact then leads to the conclusion:

To every closed convex analytic curve without rectilinear segments, at least one square may be inscribed.†

In what follows I shall show that at least one square may be inscribed in any convex rectilinear polygon, and generally in any closed convex curve formed by analytic arcs.

When nothing to the contrary is stated, all points of the boundary are supposed to be included by the domain bounded by the curve. Such a curve has the property that all points of the segment joining any two distinct points of the curve belong to the enclosed domain, and that no other points of the domain lie on the straight line joining the two points.‡ Excluding for the present certain cases where pairs of rectilinear segments of the curve become parallel, two medians M_σ and M_τ, as defined above, accordingly can not have more than one point of intersection. This is also true in all cases of pairs of

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† The problem connected with this theorem was suggested to me by Dr. Kempner. After I had found a proof of the theorem and sent it to this Journal for publication, I was told that Dr. Toeplitz (at that time in Göttingen) and his pupils had succeeded in solving the problem. A statement to this effect, without proof, may be found in the *Verhandlungen der schweizerischen Naturforschenden Gesellschaft in Solothurn*, August 1, 1911, p. 197, according to which Dr. Toeplitz succeeded in proving the theorem for convex curves only. I have not been able to find a publication of his method and results.

‡ Minkowsky, "Theorie der konvexen Körper," zweiter Band, p. 154.

parallel sides, when the distance between two parallel sides is less than the smaller of the sides. If there were more than one point of intersection, it would be possible to inscribe at least two rhombs in the curve with parallel diagonals. Among the eight vertices of two rhombs of this kind there would, however, be at least four that would form a reentrant quadrangle, which is against the assumption of a convex curve. The case of pairs of parallel sides will be considered later.

§ 2. MEDIANS OF A CONVEX RECTILINEAR POLYGON.

I. *Medians Near Four Consecutive Vertices of the Polygon.*

To study the variation of the medians as the direction of the system of parallel chords changes continuously, take first a direction $\sigma_1 \parallel DB$. The median of the chords parallel to σ_1 within the triangle BCD is the line

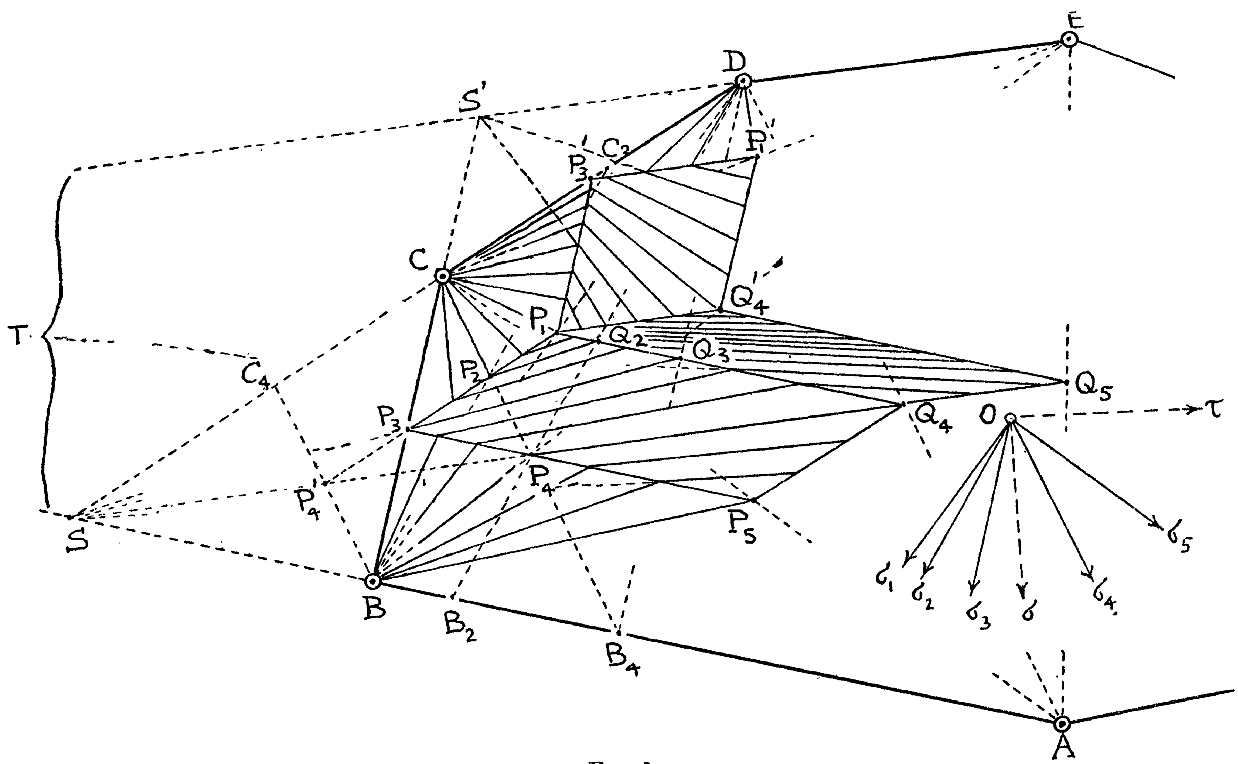


FIG. 1.

joining C with the mid-point P_1 of DB. Let σ_2 be an intermediate direction in the continuous change in the positive sense from σ_1 to $\sigma_5 \parallel CA$. Draw C_2B and $DB_2 \parallel \sigma_2$, and designate by P_2 and Q_2 their mid-points (Fig. 1). The median of all parallel chords parallel to σ_2 between C and DB_2 consists of CP_2 and P_2Q_2. For $\sigma_3 \parallel CB$, the median P_3Q_3 terminates in the mid-point P_3 of CB. As σ_2 approaches σ_3, CP_2 approaches CP_3 as a limit. For $\sigma_4 \parallel DA$,

median consists of the lines BP_4 and P_4Q_4, where Q_4 is the mid-point of 1. Turning in the opposite direction, as σ_4 approaches σ_3, BP_4 approaches BP_3 as a limit. Finally, for $\sigma_5 \parallel CA$, the median consists of the line BP_5. It will be noticed that as σ_4 approaches σ_5, P_4Q_4 approaches $P_5(0)$ as a limit. The prolongations of P_2Q_2, P_3Q_3, P_4Q_4, ... all pass through S, the intersection of DC and AB produced. Hence, as the direction of the system of parallel chords changes continuously from σ_1 to σ_5, corresponding medians, originating from C, P_3 and B, change continuously, except for $\sigma_3 \parallel CB$, where as a limit CP_3 changes abruptly into BP_3. The medians consist of polygonal lines of two sides, whose vertices P_1, ..., P_2, Q_2, ..., P_3, Q_3, ..., P_4, Q_4, ..., P_5, with the exception of the initial points C and B, are located on the sides of the parallelogram formed by the mid-points of BC, AD, DB and CA. From this fact results a simple construction of the system of medians: Construct the parallelogram $P_1P_3P_5Q_4$; draw all lines bounded by this parallelogram and whose prolongations pass through S; and connect the points on P_1P_3 with C and those on P_3P_5 with B. Within the parallelogram the medians form a continuous set.

As σ_1 changes to σ_2, P_1 moves on P_1P_3 to P_2, so that BP_1 describes a pencil parallel to the pencil of directions $(O \,.\, \sigma_1 \ldots \sigma_2)$. But there is also

$$C \,.\, P_1P_2 \ldots \barwedge B \,.\, P_1P_2 \ldots \barwedge \,.\, S \,.\, P_1P_2 \ldots,$$

and consequently

$$S \,.\, P_1P_2 \ldots \barwedge O \,.\, \sigma_1\sigma_2 \ldots,$$

or

$$S \,.\, P_1Q_2 \ldots \barwedge O \,.\, \sigma_1\sigma_2 \ldots.$$

The question is whether these projectivities are confined to the points of a side of the parallelogram only or whether they hold for the whole pencil through S contained within the parallelogram. Take, for instance, P_4, which is obtained by drawing $B_4C \parallel \sigma_4$. $BC_4 \parallel B_4C$ cuts P_1P_3 produced at P'_4. As $CP_3 = BP_3$ and $P_3P'_4 \parallel CC_4$, there is also $BP'_4 = P'_4C_4$; hence, as $CP_4 = P_4B_4$, SP'_4 produced must pass through P_4. From this follows immediately the projective relation:

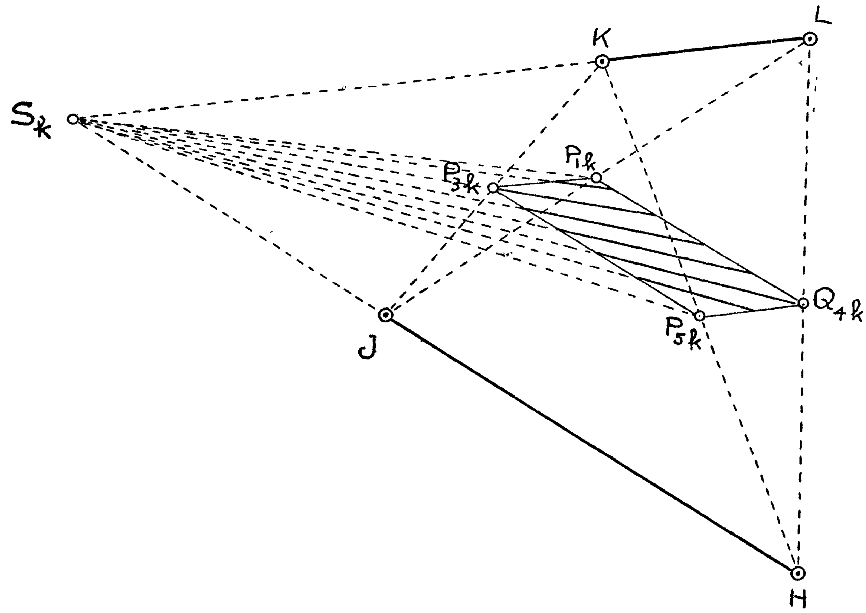
$$S \,.\, P_1P_2 \ldots P_3 \ldots P_4 \ldots P_5 \barwedge O \,.\, \sigma_1\sigma_2 \ldots \sigma_3 \ldots \sigma_4 \ldots \sigma_5.$$

Considering next the quadrangle $BCDE$, the system of medians associated with it may be constructed in precisely the same manner as in case of $ABCD$. The prolonged portions of the medians within the parallelogram $P'_1P'_3P_1Q'_4$ all pass through S', the intersection of ED and BC produced, while the remaining portions are obtained by joining the points of P'_3P_1 to C and those

of $P'_3 P'_1$ to D. From this it is seen that the parts of all medians originating at any vertex, like C, form a pencil whose rays completely fill the parallelogram $P'_3 C P_3 P_1$, and that consequently *no two medians of the complete system of medians associated with the given polygon can intersect within the parallelograms determined by half the sides adjacent at any vertex.*

II. *Medians Associated with a Quadrangle Determined by Any Two Non-adjacent Sides of the Polygon.*

Let $HJKL$ be such a quadrangle (Fig. 2). In perfect analogy with the discussion of the medians within the parallelogram $P_1 P_3 P_5 Q_4$ (Fig. 1), it is found that as the direction of the system of parallel chords changes continu-



FIG, 2.

ously in the positive sense from LJ to KH, the corresponding medians form a pencil of rays through S_k, the intersection of LK and HJ, within the parallelogram $P_{1k} P_{3k} P_{5k} Q_{4k}$, such that the pencil through S_k is projective to the pencil of corresponding directions. Thus, in Fig. 1, the continuation of the medians terminating in points of $P_1 Q_4$ and $P_1 Q'_4$ are obtained by completing the parallelogram $P_1 Q_4 Q_5 Q'_4$ associated with the quadrangle $ABDE$, and connecting the extremities of the previously constructed medians on $P_1 Q_4$ and $P_1 Q'_4$ with the intersection T of AB and ED. The rays of this pencil within the parallelogram $P_1 Q_4 Q_5 Q'_4$ are the continued medians.

In a polygon of n sides we can now first form n parallelograms, like $P_1 P_3 P_5 Q_4$, having one of their vertices at the mid-points of the sides. Then,

with every side are associated $(n-5)$ parallelograms as defined under II, so that there are $\frac{n(n-5)}{2}$ parallelograms of this kind. Altogether there are $n+\frac{n(n-5)}{2}=\frac{n(n-3)}{2}$ parallelograms on whose sides the vertices of the polygonal lines forming the medians are located. The sides of these parallelograms are parallel to the sides of the given polygon. In every triangle with one side of the polygon as a base and any two diagonals from the extremities of the base to a vertex of the polygon as the other two sides, the line (parallel to the base) joining the mid-points of these two sides is common to two, but only two, parallelograms of the system. Thus, in the triangle DAB (Fig. 1), P_1Q_4 is common to the parallelograms $P_1Q_4P_5P_3$ and $P_1Q_4Q_5Q'_4$ associated with the quadrangles $ABCD$ and $ABDE$. Every point of P_1Q_4 therefore belongs to one and only one median of the entire system of medians. These results may be stated in

THEOREM 1. *In the foregoing system of parallelograms enclosing pencils of medians, every side, except those terminating in mid-points of sides of the polygon, is common to two parallelograms only. Through every point of these common sides only one median of the entire system of medians passes.*

Associated with each of these parallelograms is a point S, or T, through which the prolonged medians within the parallelogram pass. The sides of the n parallelograms terminating at the mid-points of the n sides of the given polygon form a polygon of $2n$ points bounding a closed domain (P) within which the $\frac{n(n-3)}{2}$ parallelograms are located. Within each of these parallelograms the medians associated with it, and prolonged, form a pencil which is projective with the pencil of corresponding directions. These results may be stated in

THEOREM 2. *Within the domain (P), the system of medians of a closed convex rectilinear polygon associated with the continuous set of all directions σ through a fixed point (pencil) is continuous with this pencil. The directions of the medians within each parallelogram, and associated with it, form a pencil which is projective with the pencil of corresponding directions σ. Within the domain of the given polygon (excluding the points of the boundary), and exterior to the domain (P), no medians can intersect. Within the domain (P), the system of medians is continuous.*

So far we have assumed that no pair of sides of the polygon are parallel. From the foregoing development it is obvious that with every quadrangle

determined by two distinct sides of the polygon is associated uniformly a parallelogram whose sides are parallel in pairs to the two sides considered, and which is continuously filled with the segments of a pencil of medians.

In Fig. 1, with the sides AB and DE, for example, is associated the parallelogram $P_1 Q_4 Q_5 Q_4'$. When $AB \parallel DE$, the parallelogram degenerates into a single line $P_1 Q_5$, and the medians of the parallelogram, as AB becomes parallel to DE, will be transformed also into a continuous set on the segment $P_1 Q_5$, varying from 0 at P_1 to the length of $P_1 Q_5$ and from this again to 0 at Q_5. From this it is seen that continuity of the system of medians with respect to both magnitude and position is maintained also in case of pairs of parallel sides.

Theorem 2 is therefore also valid in this case.

§ 3. INSCRIBED RHOMBS.

Consider again a closed convex polygon with no pair of parallel sides and two orthogonal directions σ and τ through O, and construct the medians M_σ and M_τ associated with them. The extremities A_σ and Z_σ of M_σ, and A_τ and Z_τ of M_τ always lie on pairs of opposite sides of a rectangle. M_σ and M_τ therefore necessarily intersect. As stated in the introduction, for a convex curve there can be only one point of intersection R, which, according to theorem 1, lies within the domain (P). As the pair $(\sigma\tau)$ of orthogonal directions changes continuously, the medians M_σ and M_τ within (P) change continuously also, and consequently their point of intersection R describes a continuous curve.

In the neighborhood of R the medians $M_\sigma M_{\sigma'} M_{\sigma''} \ldots\ldots$ and $M_\tau M_{\tau'} M_{\tau''} \ldots\ldots$ form pencils, so that

$$M_\sigma M_{\sigma'} M_{\sigma''} \ldots\ldots \barwedge O \,.\, \sigma\sigma'\sigma'' \ldots\ldots,$$

and

$$M_\tau M_{\tau'} M_{\tau''} \ldots\ldots \barwedge O \,.\, \tau\tau'\tau'' \ldots\ldots;$$

and as, obviously,

$$O \,.\, \sigma\sigma'\sigma'' \ldots\ldots \barwedge O \,.\, \tau\tau'\tau'' \ldots\ldots,$$

there is also

$$M_\sigma M_{\sigma'} M_{\sigma''} \ldots\ldots \barwedge M_\tau M_{\tau'} M_{\tau''} \ldots\ldots.$$

In the neighborhood of the point considered, the curve described by R is therefore an arc of a conic.

Now turning σ and τ continuously through a right angle, the two directions and consequently the two medians M_σ and M_τ are interchanged by a continuous process. R describes therefore a closed curve. We have, therefore:

THEOREM 3. *The locus of the points of intersection R of the medians M_σ and M_τ associated with all pairs of orthogonal directions is a closed continuous curve composed of arcs of conics.*

The case remains to be investigated where one or more pairs of sides of the polygon each consist of parallel sides. We may limit ourselves to one pair, since the effect on the medians by any other pair will be of the same nature.

As stated before, all portions of medians determined by two parallel sides lie in the mid-line between the two sides and vary continuously in their length

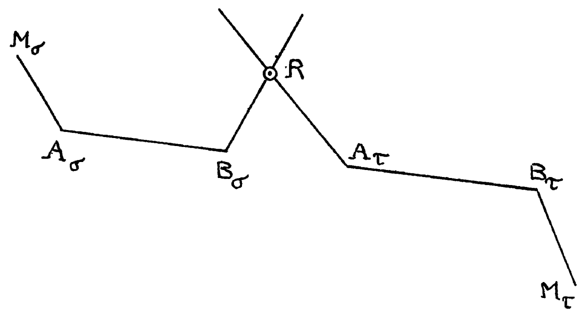


FIG. 3.

and extremities. Consequently, if we take two medians M_σ and M_τ corresponding to a pair of orthogonal directions (σ, τ) and consider the portions $A_\sigma B_\sigma$ and $A_\tau B_\tau$ of each determined by the two parallel sides, two cases may occur:

1) For all pairs (σ, τ) the segments $A_\sigma B_\sigma$ and $A_\tau B_\tau$ have no point in common (Fig. 3), so that M_σ and M_τ have in every position one and only one definite point of intersection R. As can easily be established by elementary geometry, this will, for instance, always be the case when the distance between the two parallel sides of the polygon is less than the smaller of the two sides.

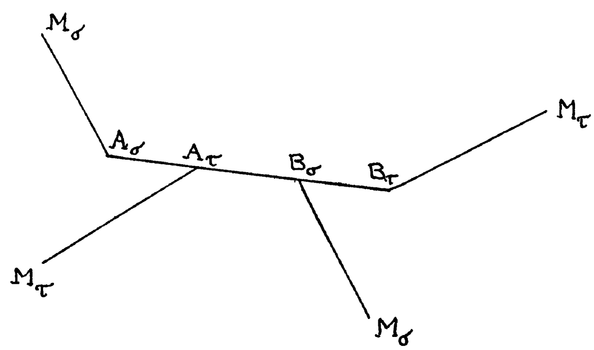


FIG. 4.

Every position of R is the center of an inscribed rhomb. As σ changes continuously, the system of chords also changes continuously. Consequently, as R describes a continuous curve, the corresponding inscribed rhombs form a continuous set as to magnitude and position.

2) For all pairs (σ, τ) of a certain continuous set S, the segments $A_\sigma B_\sigma$ and $A_\tau B_\tau$ have a portion $A_\tau B_\sigma$ in common (Fig. 4). This means that for

every pair (σ, τ) of S there is an infinite number of inscribed rhombs whose centers form the continuous set of points on the segment $A_\tau B_\sigma$, and whose axes are parallel to the directions σ and τ. Their vertices lie in pairs on the two parallel sides. But as σ and τ within S change continuously, also the extremities A_τ and B_σ, and consequently the segment $A_\tau B_\sigma$ and with it the infinite set of rhombs, including those with centers at A_τ and B_σ, associated with every pair (σ, τ) of the system S, change continuously. For the extreme pairs (σ, τ) of the system S the segment $A_\tau B_\sigma$ reduces to a point and is continuously connected with the points of the curve traced by the uniform intersections of all pairs M_σ and M_τ belonging to the remaining set (σ, τ) outside of S.

Evidently, when M_σ and M_τ have a common segment $A_\tau B_\sigma$, there are not other intersections of the two medians outside of the segment. Otherwise we could construct four points forming a reentrant quadrangle, which is not possible in a convex curve. Hence also in the second case the whole set of inscribed rhombs is continuous, and we have:

THEOREM 4. *The set of rhombs inscribed in a convex rectilinear polygon is continuous.*

Turning σ and τ through a right angle and designating the inscribed rhomb in the initial position by $A_1 A_2 A_3 A_4$, then, after the interchange of σ and τ, the original rhomb, in the same order, will have continuously changed into a rhomb $A'_1 A'_2 A'_3 A'_4 \equiv A_2 A_3 A_4 A_1$. The smaller diagonal, say $A_1 A_3$, changes into the larger, $A_2 A_4$, while simultaneously the larger, $A_2 A_4$, passes into the smaller, in both cases through a continuous set of diagonals. Hence, we have exactly the same situation as in case of an oval (*loc. cit.*), and we can state

THEOREM 5. *In every convex rectilinear polygon, at least one square may be inscribed.*

§ 4. CONVEX CURVE FORMED BY ANALYTIC ARCS.

The foregoing construction is evidently valid for a convex polygon of any number of rectilinear sides.

Suppose now that a convex polygon formed by analytic arcs (including rectilinear segments) be given. Replace all non-rectilinear arcs by polygonal lines of any number of sides inscribed in these arcs, so that for the complete rectilinear convex polygon obtained in this manner the foregoing constructions may be applied. They hold no matter how small the sides of the polygonal lines inscribed in the arcs finally become. Increasing the number of sides and at the same time decreasing their length indefinitely, the

inscribed polygon will approach the given convex curve formed by analytic arcs as a limit. If σ is any direction, then any line l parallel to σ cutting the curve in two points C_1 and C_2 cuts the polygon in general in two points P_1 and P_2 (Fig. 5). Designating the mid-point of C_1C_2 by C and that of P_1P_2 by P, choosing on l any point O, so that

$$OC_1 = a_1, \quad OC_2 = a_2, \quad OP_1 = b_1, \quad OP_2 = b_2,$$

then

$$OC = \tfrac{1}{2}(a_1 + a_2), \quad OP = \tfrac{1}{2}(b_1 + b_2),$$

and

$$PC = OC - OP = \tfrac{1}{2}[(a_1 - b_1) + (a_2 - b_2)].$$

The side of the inscribed polygon on which P_1 lies approaches the arc in which it is inscribed (as a chord) as a limit, so that $OC_1 - OP_1 = a_1 - b_1 = \varepsilon_1$

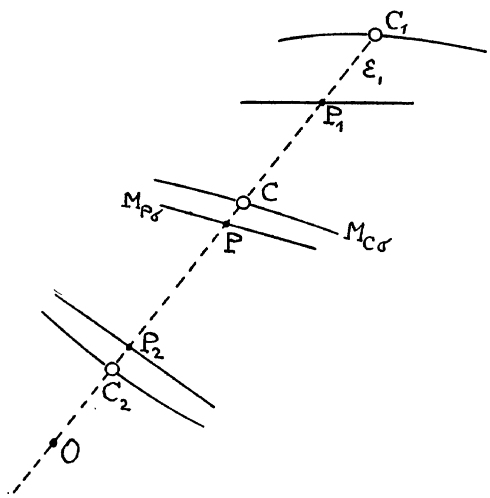


FIG. 5.

can be made as small as we please. Likewise $OC_2 - OP_2 = a_2 - b_2 = \varepsilon_2$ can be made arbitrarily small. But C is a point of the median associated with σ and the given curve. Consequently, as ε_1 and ε_2 become arbitrarily small, *i. e.*, as the inscribed polygon approaches the given curve as a limit, also

$$PC = \tfrac{1}{2}(\varepsilon_1 + \varepsilon_2)$$

becomes arbitrarily small. As P is a point of the median associated with the inscribed polygon, we conclude from this that as the inscribed polygon approaches the given curve as a limit, the median $M_{P\sigma}$ associated with the inscribed polygon approaches the median $M_{C\sigma}$ associated with the given curve as a limit. Passing to the limit does not destroy any of the properties of continuity and uniformity of the intersections of $M_{P\sigma}$ and $M_{P\tau}$ while passing into $M_{C\sigma}$ and $M_{C\tau}$. The domain (P) approaches a definite domain (P') as a

limit. Outside of (P') no two medians intersect, and within (P') the set of medians is continuous. By the same argument that led to theorem 5, we conclude:

THEOREM 6. *It is always possible to inscribe at least one square in any convex curve formed by analytic arcs.*

An interesting special case is obtained when the curve is symmetric with respect to a fixed center. All medians now pass through the center; the locus of the R's coincides with the center. Hence:

THEOREM 7. *The center of a square inscribed in a convex curve formed by analytic arcs, and with central symmetry, coincides with the center of the curve.*